Calculus in Context

The Five College Calculus Project

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Preface: 2008 edition

We are publishing this edition of *Calculus in Context* online to make it freely available to all users. It is essentially unchanged from the 1994 edition.

The continuing support of Five Colleges, Inc., and especially of the Five College Coordinator, Lorna Peterson, has been crucial in paving the way for this new edition. We also wish to thank the many colleagues who have shared with us their experiences in using the book over the last twenty years—and have provided us with corrections to the text.

Preface: 1994 edition

Our point of view We believe that calculus can be for our students what it was for Euler and the Bernoullis: A language and a tool for exploring the whole fabric of science. We also believe that much of the mathematical depth and vitality of calculus lies in these connections to the other sciences. The mathematical questions that arise are compelling in part because the answers matter to other disciplines as well.

The calculus curriculum that this book represents started with a "clean slate;" we made no presumptive commitment to any aspect of the traditional course. In developing the curriculum, we found it helpful to spell out our **starting points**, our **curricular goals**, our **functional goals**, and our view of the **impact of technology**. Our starting points are a summary of what calculus is really about. Our curricular goals are what we aim to convey about the subject in the course. Our functional goals describe the attitudes and behaviors we hope our students will adopt in using calculus to approach scientific and mathematical questions. We emphasize that what is missing from these lists is as significant as what appears. In particular, we did *not* not begin by asking what parts of the traditional course to include or discard.

Starting Points

- Calculus is fundamentally a way of dealing with functional relationships that occur in scientific and mathematical contexts. The techniques of calculus must be subordinate to an overall view of the underlying questions.
- Technology radically enlarges the range of questions we can explore and the ways we can answer them. Computers and graphing calculators are much more than tools for teaching the traditional calculus.

Starting Points—continued

- The concept of a dynamical system is central to science Therefore, differential equations belong at the center of calculus, and technology makes this possible *at the introductory level*.
- The process of successive approximation is a key tool of calculus, even when the outcome of the process—the limit—cannot be explicitly given in closed form.

Curricular Goals

- Develop calculus in the context of scientific and mathematical questions.
- Treat systems of differential equations as fundamental objects of study.
- Construct and analyze mathematical models.
- Use the method of successive approximations to define and solve problems.
- Develop geometric visualization with hand-drawn and computer graphics.
- Give numerical methods a more central role.

Functional Goals

- Encourage collaborative work.
- Empower students to use calculus as a language and a tool.
- Make students comfortable tackling large, messy, ill-defined problems.
- Foster an experimental attitude towards mathematics.
- Help students appreciate the value of approximate solutions.
- Develop the sense that understanding concepts arises out of working on problems, not simply from reading the text and imitating its techniques.

Impact of Technology

- Differential equations can now be solved numerically, so they can take their rightful place in the introductory calculus course.
- The ability to handle data and perform many computations allows us to explore examples containing more of the messiness of real problems.
- As a consequence, we can now deal with credible models, and the role of modelling becomes much more central to our subject.

Impact of Technology—continued

- In particular, introductory calculus (and linear algebra) now have something more substantial to offer to life and social scientists, as well as to physical scientists, engineers and mathematicians.
- The distinction between pure and applied mathematics becomes even less clear (or useful) than it may have been.

By studying the text you can see, quite explicitly, how we have pursued the curricular goals. In particular, every one of those goals is addressed within the very first chapter. It begins with questions about describing and analyzing the spread of a contagious disease. A model is built, and the model is a system of coupled non-linear differential equations. We then begin a numerical assault on those equations, and the door is opened to a solution by successive approximations.

Our implementation of the functional goals is less obvious, but it is still evident. For instance, the text has many more words than the traditional calculus book—it is a book to be read. Also, the exercises make unusual demands on students. Most exercises are not just variants of examples that have been worked in the text. In fact, the text has rather few simple "template" examples.

Shifts in Emphasis It will also become apparent to you that the text reflects substantial shifts in emphasis in comparison to the traditional course. Here are some of the most striking:

HOW THE EMPHASIS SHIFTS:	
INCREASE	DECREASE
concepts	techniques
geometry	algebra
graphs	formulas
brute force	elegance
numerical solutions	closed-form solutions

Euler's method is a good example of what we mean by "brute force." It is a general method of wide applicability. Of course when we use it to solve a differential equation like $y'(t) = t$, we are using a sledgehammer to crack a peanut. But at least the sledgehammer *does* work. Moreover, it works with coconuts (like $y' = y(1-y/10)$), and it will just as happily knock down a house (like $y' = \cos^2(t)$). Of course, students also see the elegant special methods that can be invoked to solve $y' = t$ and $y' = y(1 - y/10)$ (separation of variables and partial fractions are discussed in chapter 11), but they understand that they are fortunate indeed when a real problem will succumb to these special methods.

Audience Our curriculum is not aimed at a special clientele. On the contrary, we think that calculus is one of the great bonds that unifies science, and all students should have an opportunity to see how the language and tools of calculus help forge that bond. We emphasize, though, that this is not a "service" course or calculus "with applications," but rather a course rich in mathematical ideas that will serve all students well, including mathematics majors. The student population in the first semester course is especially diverse. In fact, since many students take only one semester, we have aimed to make the first six chapters stand alone as a reasonably complete course. In particular, we have tried to present contexts that would be more or less broadly accessible. The emphasis on the physical sciences is clearly greater in the later chapters; this is deliberate. By the second semester, our students have gained skill and insight that allows them to tackle this added complexity.

Handbook for Instructors Working toward our curricular and functional goals has stretched us as well as our students. Teaching in this style is substantially different from the calculus courses most of us have learned from and taught in the past. Therefore we have prepared a handbook based on our experiences and those of colleagues at other schools. We urge prospective instructors to consult it.

Origins The Five College Calculus Project has a singular history. It begins almost thirty years ago, when the Five Colleges were only Four: Amherst, Mount Holyoke, Smith, and the large Amherst campus of the University of Massachusetts. These four resolved to create a new institution which would be a site for educational innovation at the undergraduate level; by 1970, Hampshire College was enrolling students and enlisting faculty.

Early in their academic careers, Hampshire students grapple with primary sources in all fields—in economics and ecology, as well as in history and literature. And journal articles don't shelter their readers from home truths: if a mathematical argument is needed, it is used. In this way, students in the life and social sciences found, sometimes to their surprise and dismay, that they needed to know calculus if they were to master their chosen fields. However, the calculus they needed was not, by and large, the calculus that was actually being taught. The journal articles dealt directly with the relation between quantities and their rates of change—in other words, with differential equations.

Confronted with a clear need, those students asked for help. By the mid-1970s, Michael Sutherland and Kenneth Hoffman were teaching a course for those students. The core of the course was calculus, but calculus as it is *used* in contemporary science. Mathematical ideas and techniques grew out of scientific questions. Given a process, students had to recast it as a model; most often, the model was a set of differential equations. To solve the differential equations, they used numerical methods implemented on a computer.

The course evolved and prospered quietly at Hampshire. More than a decade passed before several of us at the other four institutions paid some attention to it. We liked its fundamental premise, that differential equations belong at the center of calculus. What astounded us, though, was the revelation that differential equations could really *be* at the center—thanks to the use of computers.

This book is the result of our efforts to translate the Hampshire course for a wider audience. The typical student in calculus has not been driven to study calculus in order to come to grips with his or her own scientific questions—as those pioneering students had. If calculus is to emerge organically in the minds of the larger student population, a way must be found to involve that population in a spectrum of scientific and mathematical questions. Hence, calculus *in context*. Moreover, those contexts must be understandable to students with no special scientific training, and the mathematical issues they raise must lead to the central ideas of the calculus—to differential equations, in fact.

Coincidentally, the country turned its attention to the undergraduate science curriculum, and it focused on the calculus course. The National Science Foundation created a program to support calculus curriculum development. To carry out our plans we requested funds for a five-year project; we were fortunate to receive the only multi-year curriculum development grant awarded in the first year of the NSF program. This text is the outcome of our effort.